RODOLFO GUIDOBONO - CURSO TEORÍA CUÁNTICA DE CAMPOS JAVIER GARCÍA - EJERCICIO CAP. 43

29/10/20

(1) Demostrar que a) $\bar{u}_i u_j = \delta_{ij}$

b) $\bar{v}_i v_j = -\delta_{ij}$

$$u_i = \begin{pmatrix} ch\, \eta/2 \\ (-1)^{i-1} sh\, \eta/2 \end{pmatrix} \otimes \chi_i \qquad \chi_i \text{ puede ser } \begin{cases} \chi_+ \\ \chi_- \end{cases}$$

$\bar{u}_i = u_i^{\dagger} . \gamma_0$ donde $\gamma_0 = \sigma^3 \otimes \mathbb{I}$. $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

cambio índices para que no se confundan con el nº imaginario

$$\bar{u}_j . u_k = u_j^{\dagger} (\sigma^3 \otimes \mathbb{I}) u_k = u_j^{\dagger} (\sigma^3 \otimes \mathbb{I}) \begin{pmatrix} ch\, \eta/2 \\ (-1)^{k-1} sh\, \eta/2 \end{pmatrix} \otimes \chi_k$$

$$= u_j^{\dagger} \left[\sigma_3 \begin{pmatrix} ch \\ (-1)^{k-1} sh \end{pmatrix} \otimes \chi_k \right] = u_j^{\dagger} \begin{pmatrix} ch\, \eta/2 \\ (-1)^{k} sh\, \eta/2 \end{pmatrix} \otimes \chi_k$$

$$= \left[\left(ch \;\; (-1)^{j-1} sh \right) \otimes \chi_j^{\dagger} \right] \left[\begin{pmatrix} ch\, \eta/2 \\ (-1)^{k} sh\, \eta/2 \end{pmatrix} \right] \otimes \chi_k$$

$$= \left[ch^2 \eta/2 + (-1)^{j-1} (-1)^k sh^2 \eta/2 \right] \otimes \underbrace{\left[\chi_j^{\dagger} \chi_k \right]}_{= \delta_{jk}}$$

$$= \left(ch^2 \eta/2 + \underbrace{(-1)^{j+k-1}} sh^2 \eta/2 \right) \delta_{jk}$$

$\delta_{jk} = 1$ si $j = k$
$= 0$ si $j \neq k$

si $j = k$, el exponente resulta $2j-1 \Rightarrow$ VALOR IMPAR $\Rightarrow (-1)^{j+k-1} = -1$

en consecuencia el paréntesis será igual a $(ch^2 - sh^2) = 1$

$$\boxed{\bar{u}_j . u_k = \delta_{jk}}$$

Q.E.D.

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$$v_j = \begin{pmatrix} (-1)^{j-1}\, sh\,\eta/2 \\ ch\,\eta/2 \end{pmatrix}$$

$$\bar{v}_j v_k = v_j^{+}(\sigma_3 \otimes \mathbb{I})\, v_k = v_j^{+}(\sigma^3 \otimes \mathbb{I}) \begin{pmatrix} (-1)^{k-1}\, sh\,\eta/2 \\ ch\,\eta/2 \end{pmatrix} \otimes \chi_k$$

$$= v_j^{+}\left[\sigma^3 \begin{pmatrix} (-1)^{k-1}\, sh \\ ch \end{pmatrix} \otimes \chi_k\right] = v_j^{+} \begin{pmatrix} (-1)^{k-1}\, sh\,\eta/2 \\ -ch\,\eta/2 \end{pmatrix} \otimes \chi_k$$

$$= \left[\begin{pmatrix} (-1)^{j-1}\, sh & ch \end{pmatrix} \otimes \chi_j^{+}\right]\left[\begin{pmatrix} (-1)^{k-1}\, sh \\ -ch \end{pmatrix} \otimes \chi_k\right]$$

$$= \left[(-1)^{j+k-2}\, sh^2\,\eta/2 - ch^2\,\eta/2\right] \otimes \underbrace{\chi_j^{+}\chi_k}_{\delta_{jk}}$$

$\delta^j_k \neq 0$ si $j = k$, entonces

$j + k - 2 = 2j - 2$, que es valor par, resultando

$(-1)^{j+k-2} = +1$ cuando $j = k$

$$\bar{v}_j v_k = \left(sh^2\,\eta/2 - ch^2\,\eta/2\right)\delta_{jk}$$

$$\boxed{\bar{v}_j v_k = (-1)\,\delta_{jk}}$$

Q.E.D.

2) Demostrar que

a) $u_i^+ u_j = v_i^+ v_j = \frac{E}{mc^2}\delta_{ij}$

$$u_j^+ u_k = \left[\begin{pmatrix} ch\frac{\eta}{2} & (-1)^{j-1} sh\,\eta/2 \end{pmatrix} \otimes \chi_j^+\right]\left[\begin{pmatrix} ch\,\eta/2 \\ (-1)^{k-1} sh\,\eta/2 \end{pmatrix} \otimes \chi_k\right]$$

$$= \left[\begin{pmatrix} ch & (-1)^{j-1} sh \end{pmatrix} \begin{pmatrix} ch \\ (-1)^{k-1} sh\,\eta/2 \end{pmatrix}\right] \otimes \chi_j^+ \chi_k$$

$$= \left(ch^2\,\eta/2 + (-1)^{j+k-2}\, sh^2\,\eta/2\right) \otimes \delta_{jk}$$

$\delta_{jk} \neq 0 \iff j=k \Rightarrow j+k-2 = 2j-2$ que es par $\Rightarrow$

$\Rightarrow (-1)^{j+k-2} = 1$

$$= \left(ch^2\,\eta/2 + sh^2\,\eta/2\right)\delta_{jk} = \left(\frac{1+ch\,\eta}{2} + \frac{-1+ch\,\eta}{2}\right)\delta_{jk}$$

$= ch\,\eta\,\delta_{jk}$ pero como $ch\,\eta = p_0/mc = E/mc^2$

$$\boxed{u_i^+ u_j = E/mc^2\,\delta_{ij}}$$

$$v_j^+ v_k = \left[\begin{pmatrix} (-1)^{j-1} sh\,\eta/2 & ch\,\eta/2 \end{pmatrix} \otimes \chi_j^+\right]\left[\begin{pmatrix} (-1)^{k-1} sh\,\eta/2 \\ ch\,\eta/2 \end{pmatrix} \otimes \chi_k\right]$$

$$= \left((-1)^{j+k-2}\, sh^2\frac{\eta}{2} + ch^2\frac{\eta}{2}\right)\cdot\delta_{jk}$$

$$\boxed{v_j^+ v_k = E/mc^2\,\delta_{jk}}$$ Q.E.D.

③ Demostrar que $u_i^{\dagger}(-\vec{p})\, v_j(\vec{p}) = v_i^{\dagger}(\vec{p})\, u_j(-\vec{p}) = 0$

$$u_j(p) = \begin{pmatrix} ch\,\eta/2 \\ (-1)^{j-1}\, sh\,\frac{\eta}{2} \end{pmatrix} \otimes \chi_j \qquad u_j(-p) = \begin{pmatrix} ch\,\eta/2 \\ \underbrace{(-1)(-1)^{j-1}}_{=(-1)^j}\, sh\,\eta/2 \end{pmatrix} \otimes \chi_j$$

$$v_k(p) = \begin{pmatrix} (-1)^{k-1}\, sh\,\eta/2 \\ ch\,\eta/2 \end{pmatrix} \otimes \chi_k \qquad v_k(-p) = \begin{pmatrix} (-1)^{k}\, sh\,\eta/2 \\ ch\,\eta/2 \end{pmatrix} \otimes \chi_k$$

$$u_j^{\dagger}(-\vec{p})\, v_k(\vec{p}) = \left[\left(ch\,\frac{\eta}{2} + (-1)^j\, sh\,\frac{\eta}{2}\right) \otimes \chi_j^{\dagger}\right]\left[\begin{pmatrix} (-1)^{k-1} sh\,\eta/2 \\ ch\,\eta/2 \end{pmatrix} \otimes \chi_k\right]$$

$$= \left(ch\,\frac{\eta}{2}\,(-1)^{k-1}\, sh\,\frac{\eta}{2} + (-1)^j\, sh\,\frac{\eta}{2}\, ch\,\eta/2\right) \otimes \chi_j^{\dagger}\chi_k$$

$$= \left((-1)^{k-1} + (-1)^j\right) ch\,\eta/2\; sh\,\eta/2\; \delta_{jk}$$

si $j \neq k \Rightarrow \delta_{jk} = 0$

si $j = k \rightarrow j$ es par, $k-1$ es impar

$(-1)^{k-1} + (-1)^j = (-1) + 1 = 0$

j es impar, k es par en el mismo resultado

entonces $\boxed{u_j^{\dagger}(-\vec{p})\, v_k(\vec{p}) = 0}$

$$v_j^{\dagger}(\vec{p})\, u_k(-p) = \left[\left((-1)^{j-1}\, sh\,\frac{\eta}{2} \quad ch\,\frac{\eta}{2}\right) \otimes \chi_j\right]\left[\begin{pmatrix} ch\,\eta/2 \\ (-1)^{k}\, sh\,\frac{\eta}{2} \end{pmatrix} \otimes \chi_k\right]$$

$$= \left((-1)^{j-1} + (-1)^k\right) ch\,\frac{\eta}{2}\; sh\,\frac{\eta}{2}\; \delta_{jk}$$

siguiendo el mismo razonamiento se puede que

$\boxed{v_j^{\dagger}(\vec{p})\, u_j(-\vec{p}) = 0}$ Q.E.D.

(4) Demostrar que $\displaystyle\sum_{i=1}^{2} v_i \bar{v}_i = \frac{\gamma^\mu p_\mu - mc}{2mc}$

$$\sum_{i=1}^{2} v_i \bar{v}_i = v_1\bar{v}_1 + v_2\bar{v}_2 \qquad v_1 = \begin{pmatrix} sh \\ ch \end{pmatrix} \otimes \chi_+ \qquad v_2 = \begin{pmatrix} -sh \\ ch \end{pmatrix} \otimes \chi_-$$

$$\sum v_i\bar{v}_i = \left[\begin{pmatrix} sh \\ ch \end{pmatrix} \otimes \chi_+\right]\left[(sh\ ch) \otimes \chi_+^{+}\right]\gamma^0 + \left[\begin{pmatrix} -sh \\ ch \end{pmatrix} \otimes \chi_-\right]\left[(-sh\ ch) \otimes \chi_-^{+}\right]\cdot\gamma^0$$

$$\sum v_i\bar{v}_i = \left[\begin{pmatrix} sh^2 & sh\,ch \\ ch\,sh & ch^2 \end{pmatrix} \otimes \overbrace{\chi_+\chi_+^{+}}^{\text{matriz } 2\times 2} + \begin{pmatrix} sh^2 & -sh\,ch \\ -ch\,sh & ch^2 \end{pmatrix} \otimes \chi_-\chi_-^{+}\right]\cdot\gamma^0$$

$$= \begin{pmatrix} sh^2\,(\chi_+\chi_+^{+} + \chi_-\chi_-^{+}) & sh\,ch\,(\chi_+\chi_+^{-} - \chi_-\chi_-^{+}) \\ ch\,sh\,\underbrace{(\chi_+\chi_+^{+} - \chi_-\chi_-^{+})}_{\vec{\sigma}\cdot\hat{n}} & ch^2\,\underbrace{(\chi_+\chi_+^{-} + \chi_-\chi_-^{+})}_{=\mathbb{I}_{2\times 2}} \end{pmatrix}\cdot\gamma^0$$

$$= \begin{pmatrix} sh^2\,\eta/2\ \mathbb{I} & ch\,\eta/2\ sh\,\eta/2\ \vec{\sigma}\cdot\hat{n} \\ ch\,\eta/2\ sh\,\eta/2\ \vec{\sigma}\cdot\hat{n} & ch^2\,\eta/2\ \mathbb{I} \end{pmatrix}\cdot\gamma^0$$

$$= \begin{pmatrix} \frac{-1+ch\,\eta}{2}\ \mathbb{I} & \frac{1}{2}\,sh\,\eta\ \vec{\sigma}\cdot\hat{n} \\ \frac{1}{2}\,sh\,\eta\ \vec{\sigma}\cdot\hat{n} & \frac{1+ch\,\eta}{2}\ \mathbb{I} \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (-1+ch\,\eta)\,\mathbb{I} & -sh\,\eta\,\vec{\sigma}\cdot\hat{n} \\ sh\,\eta\,\vec{\sigma}\cdot\hat{n} & (-1-ch\,\eta)\,\mathbb{I} \end{pmatrix}$$

$$= \frac{1}{2}(-\mathbb{I}_{4\times 4}) + \frac{1}{2}\,ch\,\eta\begin{pmatrix} \mathbb{I}_{2\times 2} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} + \frac{1}{2}\,sh\,\eta\,\underbrace{\begin{pmatrix} 0 & -\vec{\sigma}\cdot\hat{n} \\ \vec{\sigma}\cdot\hat{n} & 0 \end{pmatrix}}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i\sigma^2 \qquad\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes (\vec{\sigma}\cdot\hat{n})$$

$$\sum \sigma_i \bar{\sigma}_i = -\frac{1}{2}\,\mathbb{I} + \frac{1}{2}\,\mathrm{ch}\,\eta\,\gamma^0 + \frac{1}{2}\,\mathrm{sh}\,\eta\,(-i)\,\sigma^2 \otimes (\vec{\sigma}\cdot\hat{n})$$

$$\vec{p} = mc\,\mathrm{sh}\,\eta\,\hat{n} \qquad \mathrm{sh}\,\eta\,\hat{n} = \frac{\vec{p}}{mc} \qquad p_0 = mc\,\mathrm{ch}\,\eta$$

$$\sum \sigma_i \bar{\sigma}_i = -\frac{1}{2}\,\mathbb{I} + \frac{1}{2}\,\frac{p_0}{mc}\,\gamma^0 + \frac{1}{2}\,(-i)\,\sigma^2 \otimes \left(\vec{\sigma}\cdot\frac{\vec{p}}{mc}\right)$$

$$\gamma^\mu = i\,\sigma^2 \otimes \sigma^\mu$$

$$\sum \sigma_i \bar{\sigma}_i = -\frac{1}{2}\,\mathbb{I} + \frac{1}{2}\,\frac{p_0\gamma^0}{mc} - \frac{1}{2}\,\gamma^\mu\,\frac{p^\mu}{mc} \qquad p^\mu = -p_\mu$$

$$= -\frac{1}{2}\,\mathbb{I} + \frac{1}{2}\,\frac{p_0\gamma^0}{mc} + \frac{1}{2}\,\frac{\gamma^\mu p_\mu}{mc}$$

$$= \frac{1}{2mc}\Big(-mc\,\underbrace{\mathbb{I}}_{4\times 4} + \underbrace{\gamma^\mu p_\mu}_{4\times 4}\Big)$$

$$\boxed{\sum_{i=1}^{2} \sigma_i \bar{\sigma}_i = \frac{\gamma^\mu p_\mu - mc}{2mc}}$$

Q.E.D.